Lecture Plan for Next Week Math 4B

✈️ I will be traveling for a conference in Texas next week (Monday to Thursday). Here is the plan for next week's MATH 4B:

- Wednesday in-person Lectures will be replaced by online prerecorded videos.
- Remember Monday is a holiday.
- Our Friday lecture on May 31 will be in-person as usual.
- I will post information and practice for Final Exam by the end of Friday, May 31.

Lecture 19. Solution to the Homogeneous Systems and the Eigenvalue Method

Solutions of Homogeneous Systems

associated to $\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$

Consider the the **associated homogeneous equation**

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} \tag{1}$$

We expect it to have n solutions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ that are independent in some appropriate sense, and such that every solution of Eq. (1) is a linear combination of these n particular solutions.

Given n solutions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ of Eq. (1), we write

$$\mathbf{x}_j(t) = \begin{pmatrix} x_{1j}(t) \\ \vdots \\ x_{ij}(t) \\ \vdots \\ x_{nj}(t) \end{pmatrix} \tag{2}$$

Theorem 1 Principle of Superposition

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be n solutions of the homogeneous linear equation in (1) on the open interval I. If $c_1, c_2, \cdots c_n$ are constants, then the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

is also a solution of Eq. (2) on I.

Independence and General Solutions

The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \cdots, c_n$ not all zero such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0}$$

for all t in I. Otherwise, they are **linearly independent**.

Just as in the case of a single nth-order equation, there is a Wronskian determinant that tells us whether or not n given solutions of the homogeneous equation in (1) are linearly dependent. If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are such solutions, then their Wronskian is the $n \times n$ determinant

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots x_{2n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots x_{nn}(t) \end{vmatrix}$$

using the notation in (2) for the components of the solutions.

Theorem 2 Wronskians of Solutions

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are n solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I. Suppose also that $\mathbf{P}(t)$ is continuous on I. Let

$$W = W(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$$

Then

- If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly dependent on I, then $W = 0$ at every point of I.
- If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly independent on I, then $W \neq 0$ at each point of I.

  Thus there are only two possibilities for solutions of homogeneous systems: Either $W = 0$ at every point of I, or $W \neq 0$ at no point of I.

Theorem 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers $c_1, c_2, \cdots c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

for all t in I.

Example 1 In the following question, first verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\mathbf{x}; \quad \mathbf{x}_1 = e^{2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-5t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

ANS: Step 1.

For $\vec{x}_1$, LHS $= \vec{x}_1' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \overset{\checkmark}{=}$ RHS $= \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$

For $\vec{x}_2$, LHS $= \vec{x}_2' = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} \overset{\checkmark}{=}$ RHS $= \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$

Step 2. We compute the Wronskian of $\vec{x}_1$ and $\vec{x}_2$

$$W(t) = \begin{array}{cc} \vec{x}_1(t) & \vec{x}_2(t) \\ \downarrow & \downarrow \end{array}$$

$$W(t) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 9e^{-3t} - 2e^{-3t} = 7e^{-3t} \neq 0$$

So $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are linearly independent by Thm 2.

Step 3. The general solution is (by Thm 3)

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix} \Rightarrow \begin{cases} x(t) = 3c_1 e^{2t} + c_2 e^{-5t} \\ y(t) = 2c_1 e^{2t} + 3c_2 e^{-5t} \end{cases}$$

Exercise 2 Find a particular solution of the linear system that satisfies the given initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} \mathbf{x}; \quad \mathbf{x}_1 = e^{2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-5t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}, x(0) = 8, \quad y(0) = 0$$

ANS: By Example 2, we know

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}$$

Since $\vec{x}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3c_1 e^{2\cdot 0} + c_2 e^{-5\cdot 0} \\ 2c_1 e^{2\cdot 0} + 3c_2 e^{-5\cdot 0} \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + 3c_2 \end{bmatrix} \Rightarrow \begin{cases} 8 = 3c_1 + c_2 \\ 0 = 2c_1 + 3c_2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{24}{7} \\ c_2 = -\frac{16}{7} \end{cases}$$

Thus

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{72}{7} e^{2t} - \frac{16}{7} e^{-5t} \\ \frac{48}{7} e^{2t} - \frac{48}{7} e^{-5t} \end{pmatrix}$$

or, equivalently

$$\Rightarrow \begin{cases} x(t) = \frac{72}{7} e^{2t} - \frac{16}{7} e^{-5t} \\ y(t) = \frac{48}{7} e^{2t} - \frac{48}{7} e^{-5t} \end{cases}$$

The Eigenvalue Method for Homogeneous Systems

Now we will talk about the method of solving the the first-order linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

Review: Eigenvalues and Eigenvectors

Definition. Eigenvalues and Eigenvectors

The number λ is called an ***eigenvalue*** of the $n \times n$ matrix $\mathbf{A}$ provided that

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \text{characteristic eqn for } A$$

An ***eigenvector*** associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, so that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}.$$

$$\Updownarrow$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\Updownarrow$$
$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$
$$\Updownarrow$$
$$(A - \lambda I)\vec{v} = \vec{0}$$

Example 3. Find the eigenvalues and eigenvectors of the given matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

ANS: The char. eqn for A is

$$0 = |A - \lambda I| = \left| \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda) - 6 = (\lambda - 4)(\lambda + 1) - 6 = \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$$

Thus we have two distinct eigenvalues $\lambda_1 = 5$, $\lambda_2 = -2$.

Case 1. $\lambda_1 = 5$. We want to find the eigenvector assciated to $\lambda_1 = 5$. Substitute $\lambda_1 = 5$ into $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

We have

$$\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -a+2b=0 & ① \\ 3a-6b=0 & ② \end{cases}$$

Note $-3 \times ① = ②$.

We can choose $b=1$, then $a=2b=2$.

$cA\vec{v} = c\lambda\vec{v}$

$A\underline{c\vec{v}} = \lambda\underline{c\vec{v}}$

$\boxed{\text{Remark ① You can also choose } a=1, \text{ then } b=\tfrac{1}{2}}$

Then $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_1=5$.

Remark: ① Note for any constant $c \neq 0$, $c\vec{v}_1 = \begin{bmatrix} 2c \\ c \end{bmatrix}$ is also an eigenvector associated to $\lambda_1=5$. $Ac\vec{v} = \lambda c\vec{v}$

Case 2: $\lambda_2 = -2$. We solve $(A-\lambda_2 I)\vec{v}_2 = \vec{0}$

$\uparrow$ $\begin{bmatrix} a \\ b \end{bmatrix}$

We have

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 6a+2b=0 & ③ \\ 3a+b=0 & ④ \end{cases}$$

Note $③ = 2 \times ④$.

Let's choose $a=1$, then $b=-3$.

Thus $\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector corresponds to the eigenvalue $\lambda_2=-2$.

Theorem 1 Eigenvalue Solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Let λ be an eigenvalue of the (constant) coefficient matrix $\mathbf{A}$ of the first-order linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \circledast$$

If $\mathbf{v}$ is an eigenvector associated with λ, then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

is a nontrivial solution of the system.

Idea of the proof:

Assume $\vec{x}(t) = \vec{v}e^{\lambda t}$ is a solution for $\circledast$ for some $\lambda, \vec{v}$.

Then $\vec{x}'(t) = \underline{\vec{v}\lambda \cancel{e^{\lambda t}}} = A\vec{x} = \underline{A\vec{v}\cancel{e^{\lambda t}}}$

$\Rightarrow A\vec{v} = \lambda\vec{v}$

Thus λ is an eigenvalue for A and $\vec{v}$ is the corresponding eigenvector.

The Eigenvalue Method

To solve the $n \times n$ homogeneous constant-coefficient system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we have the following steps:

1. Solve the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for the matrix $\mathbf{A}$ for the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix $\mathbf{A}$.

2. Find n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ associated with these eigenvalues by solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$.

3. Note step 2 is not always possible. If it is, then we get n linearly independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \cdots, \mathbf{x}_n(t) = \mathbf{v}_n e^{\lambda_n t} \tag{1}$$

In this case the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

of these n solutions.

Case 1. Distinct Real Eigenvalues

If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real and distinct, then we substitute each of them in turn in $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ and solve for the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. In this case it can be proved that the particular solution vectors given in (1) are always linearly independent.

Example 4 Apply the eigenvalue method to find a general solution of the given system.Then use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

$$\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 2x_1 + x_2 \end{aligned} \iff \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ANS: Step 1. Find the eigenvalues for A

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

$$\Rightarrow (\lambda-4)(\lambda+1) = 0 \Rightarrow \lambda_1 = -1 \text{ and } \lambda_2 = 4.$$

Step 2. Find eigenvector.

Case 1. $\lambda_1 = -1$, we solve $(A - \lambda_1 I)\underbrace{\vec{v}_1}_{\begin{bmatrix} a \\ b \end{bmatrix}} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 2+1 & 3 \\ 2 & 1+1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 3a + 3b = 0 \\ 2a + 2b = 0 \end{cases} \Rightarrow a + b = 0$$

Let $a = 1$, $b = -1$

So $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector to $\lambda_1 = -1$.

Case 2. $\lambda_2 = 4$, we solve $(A - \lambda_2 I)\underbrace{\vec{v}_2}_{\begin{bmatrix} a \\ b \end{bmatrix}} = \vec{0}$

$$\Rightarrow \begin{bmatrix} 2-4 & 3 \\ 2 & 1-4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -2a + 3b = 0 \\ 2a - 3b = 0 \end{cases} \Rightarrow 2a - 3b = 0$$

Let $a=3$, then $b=2$.

So $\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector corresponds to $\lambda_2 = 4$.

Step 3. We have

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad \text{and}$$

$$\vec{x}_2(t) = \vec{v}_2 e^{\lambda_2 t} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}$$

Thus we have the general solution

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}$$

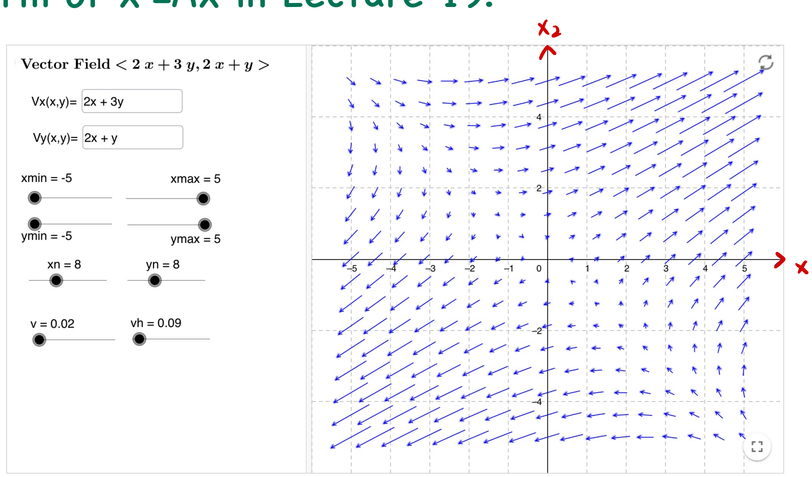
$$= \begin{bmatrix} c_1 e^{-t} + 2c_2 e^{4t} \\ -c_1 e^{-t} + 3c_2 e^{4t} \end{bmatrix}$$

Here is an online direction field calculator that we can use to generate the graph required in the question.

https://www.geogebra.org/m/QPE4PaDZ

The phase diagram is useful for analyzing the solutions in an intuitive way.

We will discuss how to obtain the direction field from the given equation in the form of x'=Ax in Lecture 19.



This applet was done thanks to the work of Linda Fahlberg-Stojanovska: https://www.geogebra.org/u/lfs-d

In this case (A has distinct eigenvalues with opposite signs), the origin $(0,0)$ is called a saddle point.

Exercise 5. Find the solution to the linear system of differential equations $\begin{cases} x' &= 4x \\ y' &= 2x + 2y \end{cases}$ satisfying the initial conditions $x(0) = 2$ and $y(0) = 5$.

Solution. We have

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

We start from finding the eigenvalues and the corresponding eigenvectors of A.

Set $|\mathbf{A} - \lambda \mathbf{I}| = 0$, we have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 4-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) = 0.$$

Thus we have $\lambda_1 = 4 \text{ and } \lambda_2 = 2.$

- Case 1. $\lambda_1 = 4$, we solve $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. We have

  $$\begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies a - b = 0$$

  So we can choose $a = 1,$ then $b = 1.$

  Thus we have $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_1 = 4.$

- Case 2. $\lambda_2 = 2$, we solve $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$. We have

  $$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 2a = 0$$

  Recall that eigenvectors cannot be zero. So we can choose $b = 1.$

  Thus we have $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_2 = 2.$

Therefore by **Theorem 1**, we have the general solution as

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So $x(t) = c_1 e^{4t}$ and $y(t) = c_1 e^{4t} + c_2 e^{2t}$.

As $x(0) = 2$ and $y(0) = 5$, we have $x(0) = c_1 = 2$ and $y(0) = c_1 + c_2 = 5$. Thus $c_1 = 2$ and $c_2 = 3$.

So $x(t) = 2e^{4t}$ and $y(t) = 2e^{4t} + 3e^{2t}$.

Exercise 6. The general solution of the linear system $\boldsymbol{y}' = A\boldsymbol{y}$ is

$$\boldsymbol{y}(t) = \begin{bmatrix} e^{t/3} & 0 \\ 0 & e^{t/7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Determine the constant coefficient matrix A.

Solution.

Rewrite the general solution

$$\boldsymbol{y}(t) = \begin{bmatrix} e^{t/3} & 0 \\ 0 & e^{t/7} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^{t/3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t/7} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Compare with the general solution form $\boldsymbol{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$

We have $\lambda_1 = 1/3$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = t/7$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So we can choose A to be the diagonal matrix

$A = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}.$

Exercise 7. Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be a solution to the system of differential equations:

$$\begin{aligned} x_1'(t) &= -4x_1(t) + 4x_2(t) \\ x_2'(t) &= \quad -20x_1(t) + 14x_2(t) \end{aligned}$$

If $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, find $\mathbf{x}(t)$.

Put the eigenvalues in ascending order when you write $x_1(t), x_2(t)$.

Solution.

We have

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -20 & 14 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Let

$$A = \begin{bmatrix} -4 & 4 \\ -20 & 14 \end{bmatrix}$$

We start from finding the eigenvalues and the corresponding eigenvectors of A.

Set $|\mathbf{A} - \lambda \mathbf{I}| = 0$, we have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda - 4 & 4 \\ -20 & -\lambda + 14 \end{vmatrix} = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) = 0$$

Thus we have $\lambda_1 = 4 \text{ and } \lambda_2 = 6$.

- Case 1. $\lambda_1 = 4$, we solve $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. We have

$$\begin{pmatrix} -8 & 4 \\ -20 & 10 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} -8a + 4b = 0 \\ -20a + 10b = 0 \end{cases}$$

  Note both equations are the same as $-2a + b = 0$. So we can choose $a = 1$, then $b = 2$.

  Thus we have $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_1 = 4$.

- Case 2. $\lambda_2 = 4$, we solve $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$. We have

$$\begin{pmatrix} -10 & 4 \\ -20 & 8 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} -10a + 4b = 0 \\ -20a + 8b = 0 \end{cases}$$

  Note both equations are the same as $-5a + 2b = 0$. Let $a = 2$, then $b = 5$.

  Thus we have $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_2 = 6$.

Therefore by **Theorem 1**, we have the general solution as

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 = c_1 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

As $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, we have

$$\mathbf{x}(0) = c_1 e^0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^0 \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + 5c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

Thus we have

$$\begin{cases} c_1 + 2c_2 = 4 \\ 2c_1 + 5c_2 = -2 \end{cases}$$

Solving this, we get

$$c_1 = 24, c_2 = -10.$$

Therefore, we have